

Holomorphic Anomalies and the Nonrenormalization Theorem

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Abstract

It has been argued that the superpotential can be renormalized in the presence of massless particles. Possible implications which have been considered include the restoration of supersymmetry at higher loops or a shift to a supersymmetric vacuum state. We argue that even in the presence of massless particles, there are no new contributions to the superpotential at any order in perturbation theory. This confirms the utility of the Wilsonian superpotential for analyzing the moduli space of the low energy theory.

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The supersymmetric nonrenormalization theorem [1] states that all perturbative corrections to the effective action come as contributions to the effective Kähler potential. In the presence of massless particles, however, it was noted in refs. [2]-[6], that certain infrared divergent contributions to the Kähler potential appear to be effectively superpotential renormalizations. An infrared divergent contribution to the Kähler potential of the following form can be rewritten as a superpotential interaction

$$\int d^4\theta \frac{D^2}{\square} g(\Phi) \sim \int d^2\theta g(\Phi) , \quad (1)$$

by using $[D^2, \bar{D}^2] \sim \square$ and integrating by parts in superspace [7] (D is the supercovariant derivative).

Some of the infrared divergent graphs that generate terms of the form (1) were explicitly calculated in refs. [3]-[6]. The simplest example is the massless Wess-Zumino model with superpotential $W = \lambda\Phi^3$. With only $\langle\Phi\Phi^\dagger\rangle$ propagators and chiral or antichiral vertices, the simplest graph that has only external Φ lines arises at two loops (fig. 1). The calculation of ref. [4] shows that it contributes an expression like eq. (1), with $g = \lambda(\lambda^\dagger\lambda)^2\Phi^3$, which renormalizes the superpotential and has a nonholomorphic dependence on the couplings.

The nonrenormalization theorem of refs. [2], [8], which forbids such nonholomorphic dependence, holds for the Wilsonian effective action, where the infrared effects are cut off, and consequently terms like (1) cannot appear. In various applications, however, when one is interested in finding the ground state of the theory, the object to minimize is the one-particle-irreducible (1PI) effective action, which is obtained from the Wilsonian effective action after integrating over all momenta, including the infrared. In the 1PI effective action, terms like (1) are allowed, and renormalization of the superpotential is possible. In theories with no massless particles, the superpotentials of the Wilsonian and 1PI effective actions coincide, and use of the Wilsonian superpotential for finding the (supersymmetric) ground state is thus justified. In theories with massless particles, however, a perturbative renormalization of the superpotential, should it exist, would contribute to the scalar potential. In particular, the renormalization of the superpotential, eq. (1), could in principle generate new terms that are consistent with the symmetries of the tree level Lagrangian, even if they are omitted from the bare action.

Such a renormalization of the superpotential, if it occurred for arbitrary values of the fields, would have important consequences for supersymmetric model building, e.g. for supersymmetric theories of flavor, or for models of supersymmetry breaking, since the vacuum could change when higher loops are included. For example, many recent theories of dynamical supersymmetry breaking employ strong gauge dynamics to break supersymmetry [9]. Below the strong coupling scale of the gauge theory, the dynamics are typically described in terms

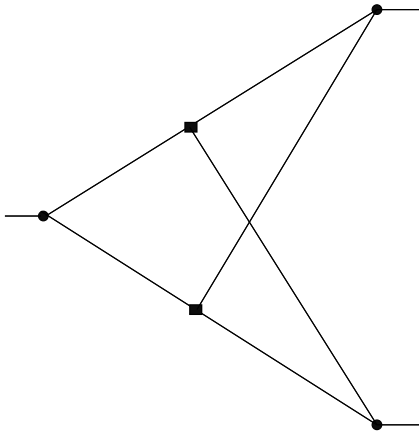


Figure 1: The lowest order graph with only chiral external lines in the massless Wess-Zumino model that generates a term of the form (1), with $g(\Phi) = \lambda(\lambda^\dagger\lambda)^2\Phi^3$. The circles and squares denote $\lambda\Phi^3$ and $\lambda^\dagger\Phi^{\dagger 3}$ vertices, respectively.

of effective, nonrenormalizable, multi-field, O’Raifeartaigh-type models. From the inconsistency of the F -term equations of motion in these effective theories, one concludes that the models break supersymmetry. The implicit assumption in this analysis is that the superpotentials of the Wilsonian and 1PI effective actions coincide. If they do, then the conclusion that supersymmetry is broken will hold in the 1PI effective action as well. If, on the other hand, the superpotential in the 1PI action is renormalized, for arbitrary field values, there exists the possibility that supersymmetry is restored after the infrared effects are taken into account, as suggested in ref. [5]. We note that a renormalization of the superpotential of the type discussed above would also contribute to a shift of the vacuum in the case of unbroken supersymmetry.

To elucidate this possibility, consider the general form of the 1PI effective potential:

$$V_{eff} = -F^*F [f_1(A^*, A) + f_2(A^*, A, F^*, F)] - [F^* w'(A^*) + F^* g'(A^*) + \text{h.c.}] , \quad (2)$$

where A, F denote the scalar and auxiliary components of the chiral superfields in the model. The function f_1 is the Kähler contribution to the effective potential, while f_2 is the non-Kählerian contribution (i.e. the one that depends on the chiral superfields as well as on their supercovariant derivatives); some non-Kählerian contributions to the effective potential have been calculated in the framework of the (supercovariant) derivative expansion, see e.g. [10] and references therein. Finally, w' denotes the gradient of the tree-level superpotential, and g' is the gradient of a possible renormalization of the superpotential due to infrared divergent contributions like the one in eq. (1). From eq. (2) one concludes that if $F = 0$ is not a solution of the tree level equations of motion, and the superpotential renormalization is absent ($g = 0$), $F = 0$ cannot be a minimum of the full 1PI effective potential as well,

and thus supersymmetry can not be restored quantum-mechanically. On the other hand, if there is superpotential renormalization, $g \neq 0$, which does not vanish at field values for which $g' \neq 0$, it is in principle possible to have an $F = 0$ minimum of the 1PI effective potential (2), and hence supersymmetry restoration.

Since we are interested in the question of supersymmetry restoration, the F terms can be nonzero. In general, the supersymmetry breaking F terms can be thought of as spurions, and one can sum over all graphs with such terms inserted. Alternatively, one can work in a background source which restores supersymmetry, and all nonsupersymmetric diagrams would be proportional to powers of the source. In general, the result of a loop diagram is quadratic in F . Note that in the presence of supersymmetry breaking, there can be logarithmic dependence on F , but the overall factor of $|F|^2$ means that these are not relevant to supersymmetry restoration (or shifting the supersymmetric vacuum). The possible exception is if there are additional holomorphic anomalies proportional to $1/F$, arising from D -terms such as

$$\int d^4\theta \frac{\Phi^\dagger}{D^2\Phi^\dagger} g(\Phi) \sim \int d^2\theta g(\Phi) . \quad (3)$$

Although we know of no such example in the literature, we also consider this possibility. The dimensional argument we give below is readily generalized to show that no holomorphic anomalies of this sort occur.

A superpotential renormalization, if it existed for field values such that $g'(A) \neq 0$, could also cause a shift in a supersymmetric vacuum. Our argument also implies the vacuum is not shifted by infrared divergent loop effects in the supersymmetric case.

In our subsequent discussion we focus on the effective action in a supersymmetry preserving (i.e. θ -independent) background. We note that even in theories with broken tree level supersymmetry such an expansion is possible, since one can always adjust the source terms so that the F -components of all background fields vanish. The effective action will then have a complicated dependence on the sources. However, we are only interested in linear terms in the sources (or, equivalently, in the limit of vanishing sources, terms linear in the F components of the background), since, as discussed above, they are the ones that are relevant for supersymmetry restoration, or for a shift of the supersymmetric vacuum. These terms can be obtained by keeping the linear term in the (supercovariant) derivative expansion; when a source is added to keep the F components of the background vanishing, this is equivalent to keeping the linear terms in the sources. We note that full dependence of the effective action on the F terms of the background field can in principle be obtained by summing up all orders in the sources (or equivalently in the derivative expansion).

We now consider the possible renormalizations to the superpotential. We will argue that when one properly analyzes the infrared structure of the theory using the observations of

Coleman and Weinberg [11], these renormalizations do not occur. Although the fields are massless at vanishing field value, nonzero field values induce masses which effectively cut off infrared divergent loops. When all the graphs are appropriately summed, there are no new contributions to the superpotential that can take nonzero value.

We consider a general effective (non-renormalizable) Wess-Zumino type theory with chiral superfields (for brevity of notation hereafter $\{\Phi\}$ denotes a generic superfield). In the background field expansion the renormalization of the superpotential (1) would arise from terms of the form:

$$\int d^4\theta \frac{D^2}{\square + |W''(\Phi)|^2} g(\{\Phi\}) . \quad (4)$$

It is clear that for general values of $|W''(\Phi)|$, there are no infrared divergences (eq. (4) is schematic and is only intended to illustrate our point: in a general background all propagators would be massive, proportional to $(\square + |W''(\Phi)|^2)^{-1}$, and the contribution of a graph like the one from fig. 1 could not in general be rewritten in this simple form). Such a term would not renormalize the superpotential (for example, for nonvanishing $W''(\Phi)$, in the zero-momentum limit, eq. (4) can be seen to contribute to the non-Kählerian term). Since a term in the superpotential should be present independent of the value of the fields, one would conclude that there are no renormalizations of the superpotential; this operator can only be interpreted as a D -term (non-Kählerian) renormalization. However, the physically relevant issue is whether there is a particular field value for which this term is responsible for supersymmetry restoration or a shift of the supersymmetric vacuum. We will now argue that this never occurs. In fact, we will show that the only place where holomorphic anomalies could have been relevant to determining the vacuum (that is, where the field-dependent mass vanishes) is precisely where the operator which would have been generated gives no contribution to the F -term of any field (that is, a sufficient number of external fields vanish).

The main idea behind the argument is that in computing the 1PI effective potential [11], one sums graphs with an arbitrary number of insertions of external lines, which provide an effective infrared cutoff on the propagators. When one turns on an arbitrary background field, all fields that couple nontrivially, and could therefore appear in a Feynman graph, obtain mass; hence in an arbitrary background there cannot be a holomorphic anomaly of the type (1). Terms like eq. (1) can only be generated for specific values of the background fields, e.g. when $W''(\Phi) = 0$ in eq. (4). The real physical issue is whether there exist field values for which the contribution to g' in eq. (2) would be nonvanishing, so that these terms are relevant for determining the vacuum of the theory (i.e. for supersymmetry restoration or a shift of the supersymmetric vacuum, as discussed above). We will argue below, that at these special field values the superpotential renormalization does not contribute to determining the vacuum of the theory.

This is easily seen for the example of the Wess-Zumino model, considered above. In order for the superpotential renormalization, $\delta W \sim \lambda(\lambda^\dagger \lambda)^2 \Phi^3$, to contribute to the vacuum energy, the F term of the field Φ has to be nonzero; hence the expectation value of Φ has to be nonvanishing. But for $\Phi \neq 0$ the propagator in the nonvanishing Φ background is massive; there is no infrared divergence and terms like (1) are consequently not generated.

It is straightforward to generalize this argument to a model with vertices of arbitrary dimension and arbitrary field content. Consider first the contribution of graphs with internal massless lines only. We wish to generate a term of the form (1), where $g(\Phi)$ has dimension 3, which can contribute to the vacuum energy, so that it is relevant, for example, for supersymmetry restoration. If two or more of the fields in $g(\Phi)$ have vanishing expectation values, the superpotential term³ (1) does not contribute to the vacuum energy, since the contribution of $g(\Phi)$ to the F -terms of all fields vanishes (unless some fields are allowed to escape to infinity; we only consider theories without classical flat directions where runaway behavior is not expected to occur). Therefore, we can allow at most one of the fields in $g(\Phi)$ to have a vanishing expectation value.

Consider first a graph with only chiral or antichiral vertices. Let the vertices that involve some external lines be of the form:

$$\frac{\{\Phi\}^k}{M^{k-3}}. \quad (5)$$

Note that if $k - 2$ of the fields Φ have nonzero expectation value, this term gives mass to the other 2 fields. Therefore, in order for the vertex (5) not to give mass to any of the internal lines when the external fields have expectation values, we can take as external at most $k - 3$ of the fields Φ , if all of them have nonvanishing expectation values, or $k - 2$ fields, if one of them has a zero expectation value.⁴ Thus the field dependence for the operator of maximum dimension, which could contribute to the energy through a holomorphic anomaly, would have the form:

$$\left(\Pi_i \frac{\{\Phi\}^{k_i-3}}{M^{k_i-3}} \right) \frac{\{\Phi\}^{l-2}}{M^{l-3}}, \quad (6)$$

where we allowed at most one external line (in the last vertex) to have zero expectation value. We now note that the term (6) has dimension 1, and since there are no mass scales anywhere in the graph, the renormalization of the superpotential (1), which could contribute to the

³It is clear that, in the framework of the loop expansion, $g(\Phi)$ is polynomial in the fields: it can not be generated at one loop, and at each higher loop order there is only a finite number of prototype graphs [11], e.g. the one from fig. 1, that generate terms like (1, 4).

⁴If the mass of a given field ϕ arises from a linear combination of, say, two vertices, it is possible that the mass could vanish for a specific choice of *nonzero* vacuum expectation values of the background fields. However, our power counting applies in this case as well: the sum of the graphs with two internal ϕ lines, with the two vertices interchanged, is proportional to the mass and vanishes.

energy, vanishes. We note that terms of the form (1) can be generated at specific field values, but do not contribute to the vacuum energy.

Above we only considered purely chiral or antichiral vertices. An effective theory will also have D -type vertices of the form:

$$\frac{\Phi^{\dagger n+1} \Phi^{m+1}}{M^{n+m}}. \quad (7)$$

Taking k of the Φ fields to be external, we find that the contribution of this vertex to the operator (6) goes like

$$\frac{\Phi^k}{M^{n+m}}. \quad (8)$$

The dimension of the operator (8) is $(k - m) - n \leq 1 - n \leq 0$, since if $k - m = 1$ one needs $n \geq 1$ in order to have at least two internal lines (so that the graph is 1PI). We thus conclude that insertions of D -type vertices cannot increase the dimension of the operator (6). Similar arguments show that D -type vertices involving supercovariant derivatives can not increase the dimension.

We conclude that graphs with only internal massless lines do not generate superpotential renormalizations of the form (1) that contribute to the energy. Therefore the nonholomorphic contributions to the superpotential generated by graphs with massless internal lines only can not lead to supersymmetry restoration or a shift of a supersymmetric vacuum.

To include possible contributions of internal massive lines as well, we note that one could potentially increase the dimension of the operator (6) if some of the internal lines, which couple to external vertices, are allowed to be massive. We do not see how including heavy states can provide the missing dimensional factor, based on the following nonrigorous reasoning.

We first note that these field dependent masses would lead to non-holomorphic field dependence: since the superfield propagators ⁵ are proportional to $(p^2 - m^\dagger m)^{-1}$, the resulting expression would depend on Φ^\dagger as well as Φ and could not be a term in the superpotential. This is true if we are trying to make up for the missing positive dimensional factors. It is possible that there is nonholomorphic field dependence in the denominator which is compensated by nonholomorphic field dependence from vertices. In this case, one can integrate out the massive lines (contract them to a point) along the lines we discuss below for a field independent mass. Then we can work in the effective theory, with D - and F -type vertices generated by integrating out the massive fields. In this effective theory, our power counting arguments above will apply.

The only remaining possibility therefore is that some of the internal lines have field independent masses (due to mass terms in the superpotential) and supply the missing dimensionful

⁵The $\langle \Phi \Phi \rangle$ -propagator at zero momentum, which goes like $1/m$, is not relevant here since in the internal lines of the 1PI graph momenta are nonvanishing, and a nonholomorphic dependence on m , and hence on the fields, would be introduced even by purely chiral propagators.

factor (a larger than 1 power of the mass) in (6). While we can not rigorously show that this does not occur, the following arguments suggest that it is unlikely:

In order to cast (6) in the form (1), one needs a factor of

$$\frac{m^2 D^2}{\square} . \quad (9)$$

However, we believe that (9) cannot be generated by massive propagators in the loops. We note that in order to obtain the contribution (9), both large and small momenta in the loops have to be relevant. In particular, to obtain a positive power of the mass, the momentum in the loop that contains the massive propagator has to be large (if the momentum is small, the corresponding contribution is suppressed by inverse powers of the mass; an effective (local) higher dimensional operator is generated, for which our previous arguments apply). The loop[s] with the massive propagator generates an effective operator of dimension 2 (i.e. a D -term). The operator, proportional to a positive power of the mass that can be produced by such loops is therefore of very low dimension. In order to embed it in an 1PI graph, the operator has to have at least two external fields; therefore by dimensional arguments the mass in the numerator has to be compensated either by mass or momentum in the denominator. If it is compensated by mass, we obtain an effective local operator for which the previous power counting applies. The only possibility then is that the loop is proportional to a factor of m^2/p^2 . In order to argue that this does not happen, recall that the singularities of a general Feynman diagram (as a function of external momenta) are determined by the solutions of the Landau equations. These equations imply that singularities in the external momenta can only occur for values of the loop momenta for which some of the propagators are on shell, while all other propagators are contracted to points—a momentum configuration corresponding to the so-called reduced diagram (obeying in addition the “physical picture” condition) [12]. Singularities for which all massive propagators in the graph are contracted to points cannot contribute terms like (9) (since they are far off-shell and are suppressed by either large momentum or large mass), while singularities that correspond to a solution of the Landau equations for which some of the massive propagators are on-shell contribute threshold factors and logarithms (see e.g. Sect. 6.3 in [13]) and are therefore irrelevant for our discussion. We thus conclude that it is unlikely that the inclusion of massive internal lines will supply the missing dimensionful factor (9) in eq. (6).

Finally, we argue, that even with broken supersymmetry (nonvanishing F terms) our arguments apply. First we note that a factor of $1/F$, eq. (3), requires vanishing fields and momenta as well; otherwise there is no singularity for zero F . One can therefore proceed as above: first contract the massive lines and consider a diagram with higher dimensional operator insertions and with massless or massive internal lines with supersymmetry breaking

masses. Because the only terms which interest us are those where the net result of evaluating the Feynman diagram must be proportional to at most one power of F , the same argument as above shows that the net dimension of the generated operator will be too low to contribute a holomorphic anomaly (again, this applies only for nonvanishing external fields so that the diagram is relevant to determining the vacuum).

We end with several comments.

First, we would like to stress the different aspects one is interested in when studying the role of the holomorphic anomaly in the superpotential renormalization (in particular its relevance for determining the vacuum), discussed in this letter, and when studying its contribution to the renormalization of the 1PI gauge kinetic function beyond one loop [2]. As discussed above, if the superpotential renormalization is to be relevant for determining the vacuum, one is necessarily interested in nonvanishing field expectation values. However, at expectation values which are relevant, the fields are massive and the holomorphic anomaly is absent. In contrast, a calculation at zero expectation values is relevant in the context of calculating the gauge beta function. Hence, when such a calculation is performed, the holomorphic anomaly contributes to the higher loop renormalization of the 1PI gauge kinetic function. Alternatively, a calculation of the gauge beta function can be performed at nonvanishing field expectation values (see [14] and references therein), when, as in the case discussed in this letter, the holomorphic anomaly contribution to the gauge kinetic function is absent and the higher loop contributions to the gauge coupling beta function are due to wave function renormalizations.

Second, we note that our analysis above assumed global supersymmetry. In supergravity, in general, both the Kähler potential and superpotential determine whether the vacuum is supersymmetric (i.e. the condition for F type breaking $W' \neq 0$ is replaced by $e^{K/2}(W' + K'W/M_{\text{Planck}}^2) \neq 0$ [7]). In this case, our analysis is only relevant for a nearly flat Kähler metric of the ultraviolet theory, at small (i.e. less than M_{Planck}) field values, where supersymmetry of the vacuum is determined only by the superpotential.

Third, we expect our conclusion that the superpotential renormalization does not contribute to the energy to hold also if the low-energy theory involves weakly coupled gauge fields, in addition to the chiral matter. The point is the same: if the gauge field couples, so that it contributes in a Feynman diagram, it will get a mass for nonvanishing field values. At the point in field space where there are superpotential terms, generated by the holomorphic anomaly [6], they would not contribute to the vacuum energy (or else, in order to contribute to the F -terms, a sufficient number of fields would have to have expectation values, making the gauge and matter fields massive, so that there would be no holomorphic anomaly). We expect the rest of the dimensional analysis argument to go through.

Finally, we note that the invariance of the Witten index does not preclude the possibility

that supersymmetry is restored when the infrared effects are taken into account (i.e. in the infinite volume limit). If a theory breaks supersymmetry, the index vanishes, however, the converse is not true: the vanishing Witten index does not imply that supersymmetry is broken in the infinite volume theory [15].

To summarize, from general considerations at arbitrary field values, we conclude that the superpotential is not renormalized. The previous arguments show furthermore that there are no new contributions to the vacuum energy due to the holomorphic anomaly, because they necessarily vanish at the point where holomorphic anomalies would have been relevant. We conclude that it is safe to address questions such as whether or not supersymmetry is broken by the vacuum using the Wilsonian superpotential, and that the only superpotential nonrenormalizations are nonperturbative.

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